ONE DIMENSIONAL FLOWS

Lecture 3: Bifurcations

3. Bifurcations

Here we show that, although the dynamics of one-dimensional systems is very limited [all solutions either settle down to a steady equilibrium or head off to $\pm\infty$] they can have an *interesting dependence on parameters*. In particular, the qualitative structure of the flow can change as parameters are varied. These qualitative changes in the dynamics are called **bifurcations** and the parameter values at which they occur are called **bifurcation points**.

Bifurcations provide models of **transitions** and **instabilities** as some **control parameter** is varied e.g. a buckling beam

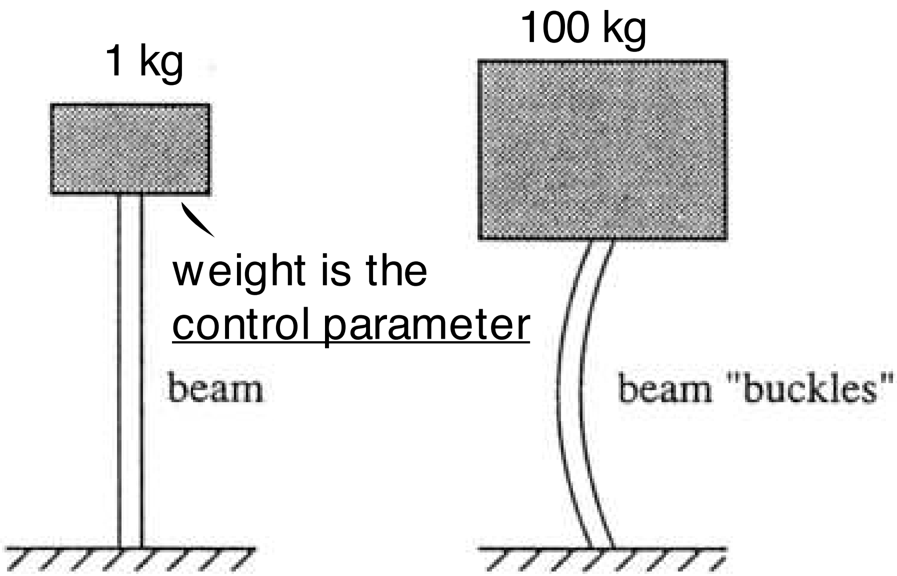


Fig. 3.0.1

3.1 Saddle-Node Bifurcation

This is the basic mechanism by which fixed points are *created and destroyed* (as some parameter is varied) e.g. $\dot{x} = r + x^2$

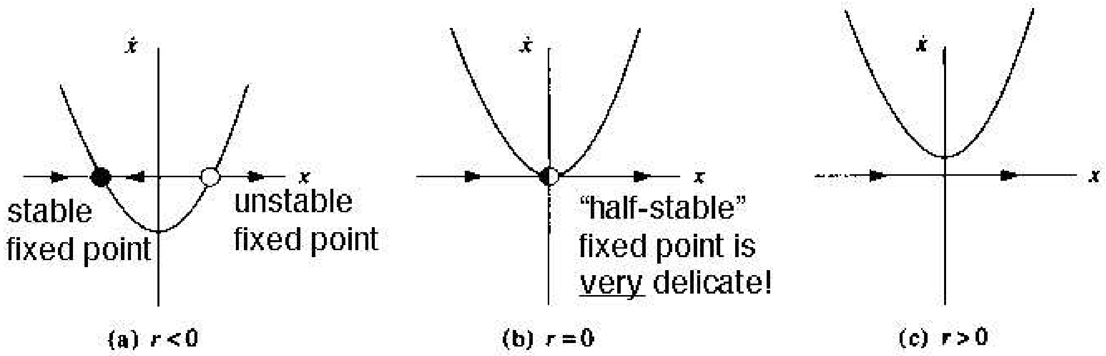


Fig. 3.1.1

$$\Downarrow$$

Bifurcation occurs at $r = 0$

Graphical conventions

There are several other ways to depict a saddle-node bifurcation

1. Stack of vector fields for discrete r values

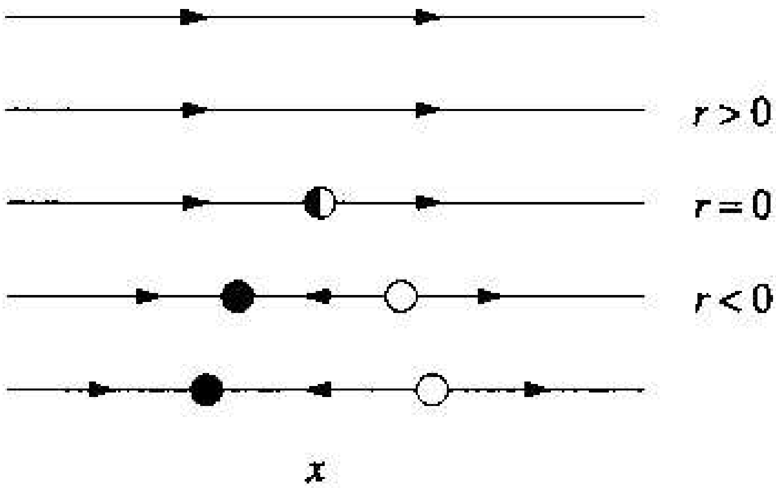


Fig. 3.1.2

2. Invert the axes of the *continuous* version of Fig. 3.1.2

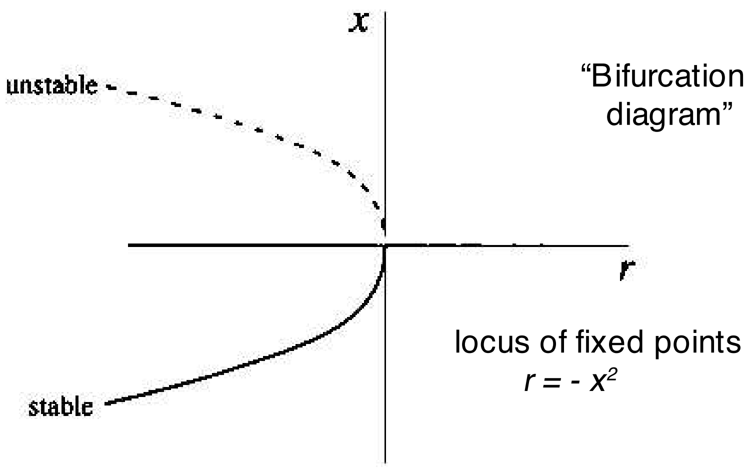


Fig. 3.1.3

NB Need starting point to be below the dashed line for system to settle into stable state

Warning about terminology

Bifurcation theory is full of conflicting terminology! The Saddle-node bifurcation is sometimes called the "fold" bifurcation, "turning point" bifurcation or "blue-sky" bifurcation (e.g. see Thompson & Stewart 2002).

Example 3.1.1 $\dot{x} = r - x^2$

- Fixed points $f(x) = r - x^2 = 0$
  $\Rightarrow x^* = \pm\sqrt{r}$

- Hence there are *two fixed points* for $r > 0$ but **none** for $r < 0$

- $f'(x^*) = -2x^* \Rightarrow x^* = \begin{cases} +\sqrt{r}: & \text{stable} \\ -\sqrt{r}: & \text{unstable} \end{cases}$

- At bifurcation points $r = 0, f'(x^*) = 0$, hence linearization vanishes when the fixed points coalesce.

Example 3.1.2 $\dot{x} = r - x - e^{-x}$

- Fixed points $f(x) = r - x - e^{-x} = 0 \Rightarrow x^* =?$

- Let's plot $r - x$ and e^{-x} and look for an intersection graphically instead...

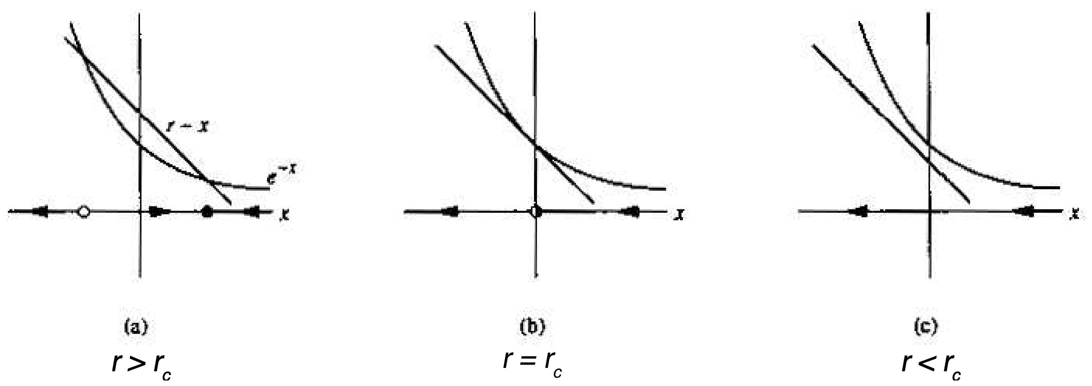


Fig. 3.1.4

$\Downarrow$

- $r - x = e^{-x}$ and $d/dx(r - x) = d/dxe^{-x}$

- Hence $r = r_c$ and bifurcation point occurs at $x = 0$.

3.2 Transcritical Bifurcation

This is the basic mechanism by which fixed points *change stability* as some parameter is varied

e.g. $\dot{x} = rx - x^2$ (cf population growth in Lecture 2)

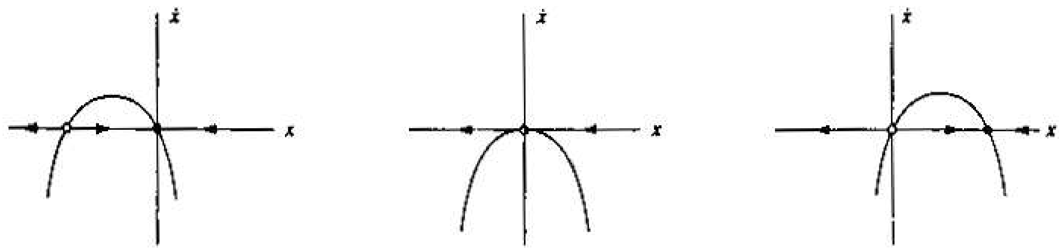


Fig. 3.2.1

"Exchange of stabilities" between $x^* = 0$ and $x^* = r$. NB unlike saddle-node case, the two fixed points don't disappear.

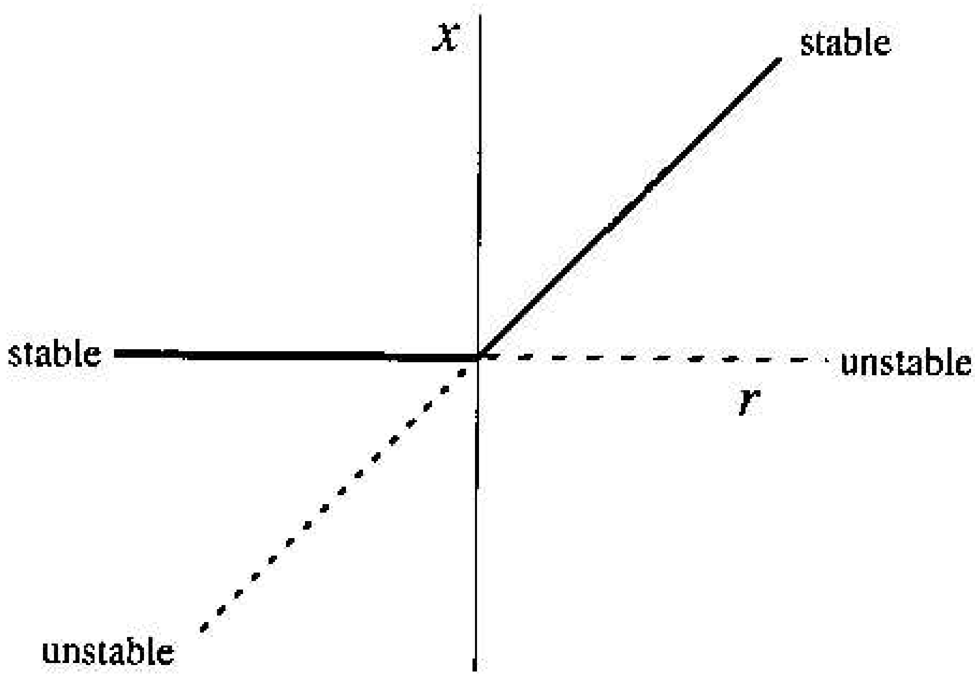


Fig. 3.2.2

Transcritical bifurcation occurs when

$$f(x^*) = f'(x^*) = 0$$

Example 3.2.1 $\dot{x} = x(1-x^2) - a(1-e^{-bx})$

By inspection $x^* = 0$ is a fixed point for all (a, b)

Expand around $x = 0$

$$\Rightarrow \dot{x} = (1-ab)x + \left(\frac{ab^2}{2}\right)x^2 + O(x^3)$$

Hence, transcritical bifurcation occurs when $ab = 1$.

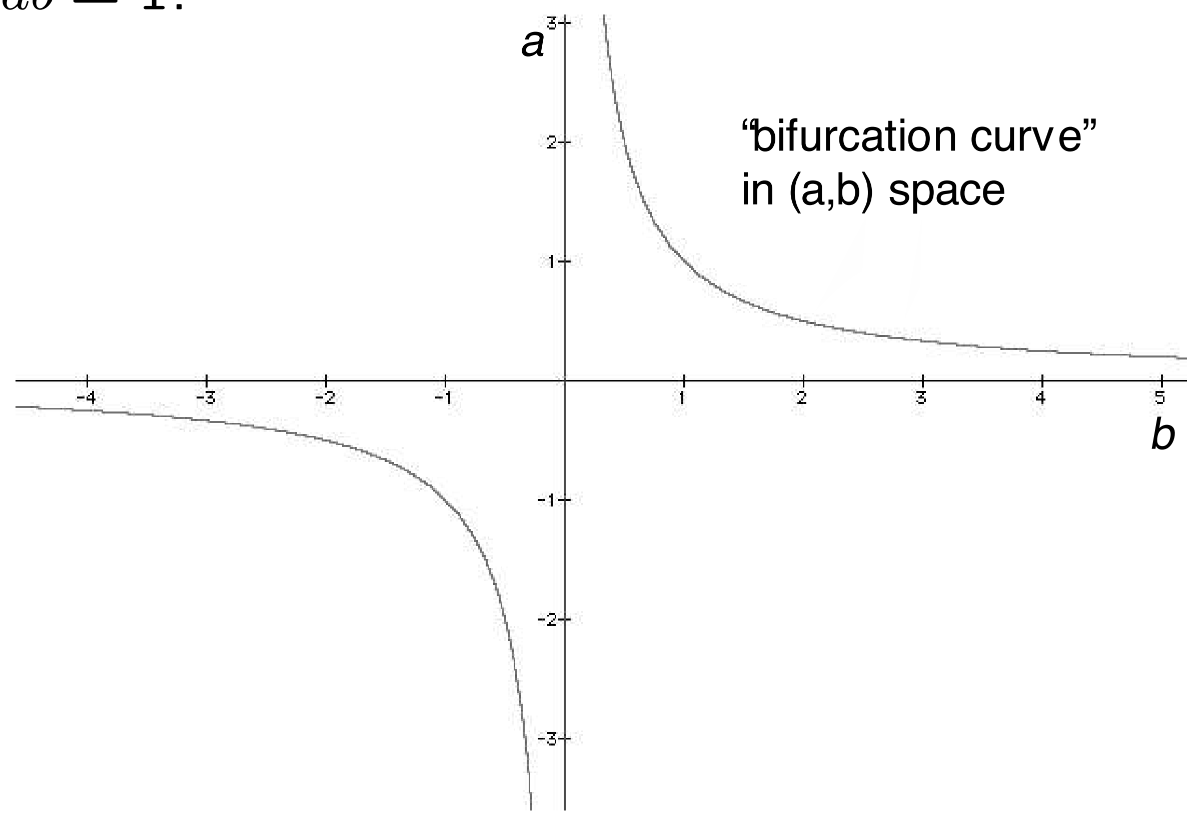


Fig. 3.2.3

Laser Threshold

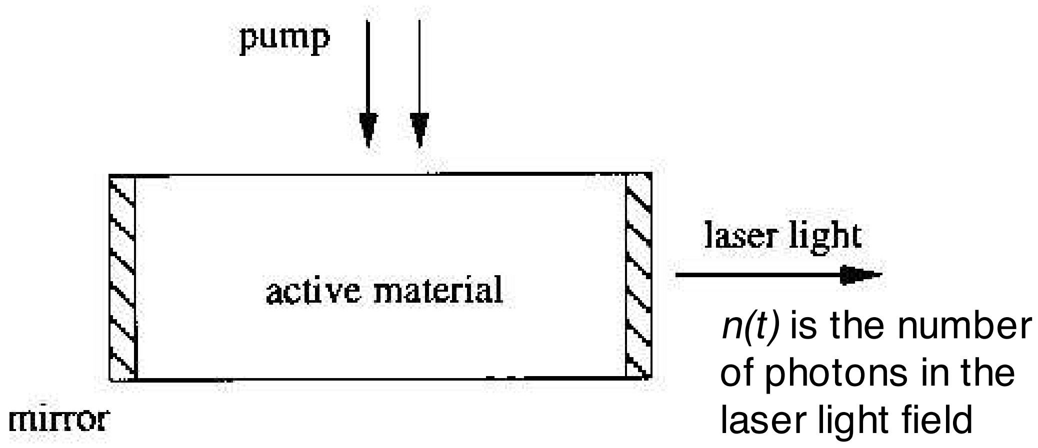


Fig. 3.3.1

From Haken (1983)

$$\begin{aligned} \dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn \end{aligned}$$

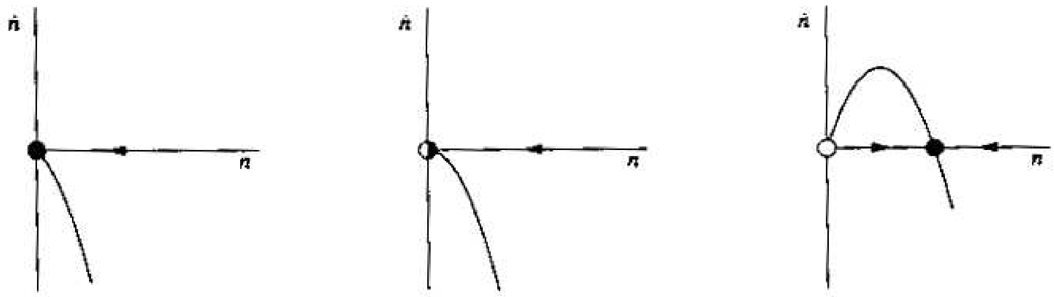
$GnN \sim$ stimulated emission; $kn \sim$ losses through endfaces; $N(t)$ is the number of excited atoms; $G > 0$ is the gain coefficient.

Also

$$N(t) = N_0 - \alpha n$$

(where N_0 depends on pump strength) since N decreases by emission of photons ($\alpha > 0$).

Hence $\dot{n} = (GN_0 - k)n - (\alpha G)n^2$



(a) $N_0 < k/G$
No stimulated emission - laser acts like a lamp!

(b) $N_0 = k/G$
transcritical bifurcation

(c) $N_0 > k/G$
spontaneous laser action

Fig. 3.3.2

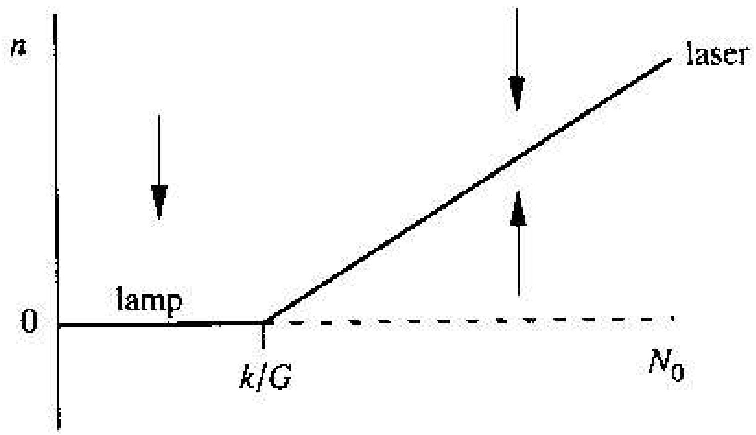


Fig. 3.3.3

3.4 Pitchfork Bifurcation

This bifurcation is common in problems that have a *symmetry*, e.g. the buckling beam, and involves fixed points appearing and disappearing in symmetrical pairs.

(i) Supercritical Pitchfork Bifurcation

e.g. $\dot{x} = rx - x^3$ [NB invariant under $x \to -x$]

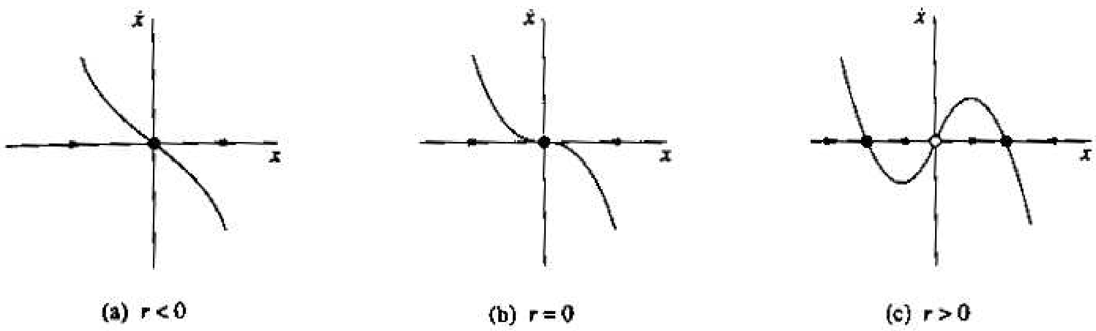


Fig. 3.4.1

- $r < 0$: Solutions decay *exponentially* fast

- $r = 0$: Linearization vanishes! Solutions decay *algebraically* fast
  $\Rightarrow$ "critical slowing down"

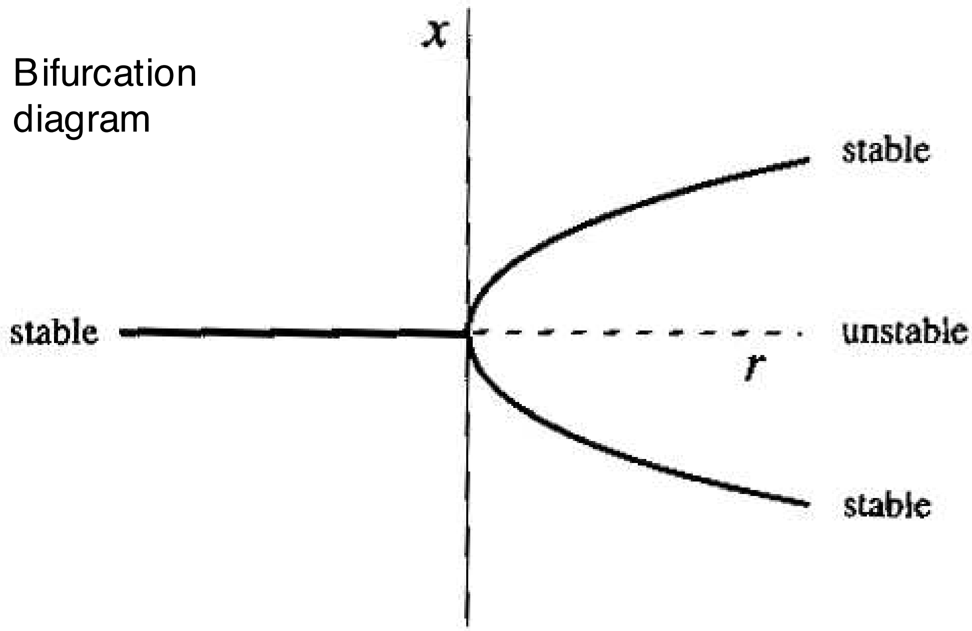


Fig. 3.4.2

... hence the term “pitchfork”

Example 3.4.1 $\dot{x} = -x + \beta \tanh x$

Arises in statistical mechanical models e.g. of magnets or neural networks

Plot $y = x$ and $y = \beta \tanh x$

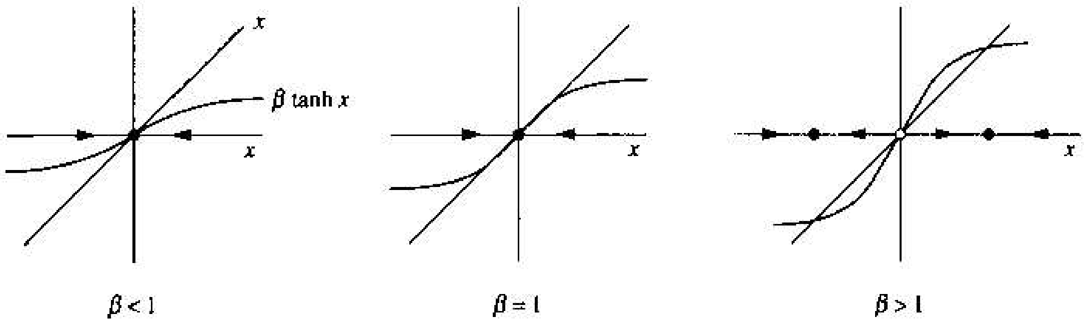


Fig. 3.4.3

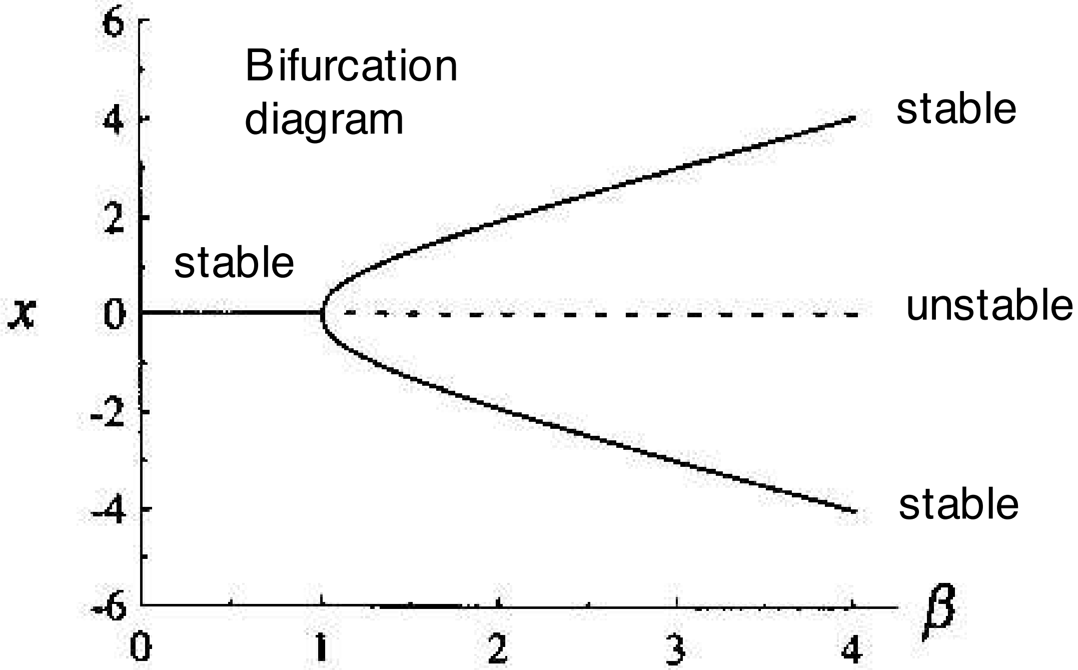


Fig. 3.4.4

(ii) Subcritical Pitchfork Bifurcation

e.g. $\dot{x} = rx + x^3$ - cubic term now *destabilizes* the system

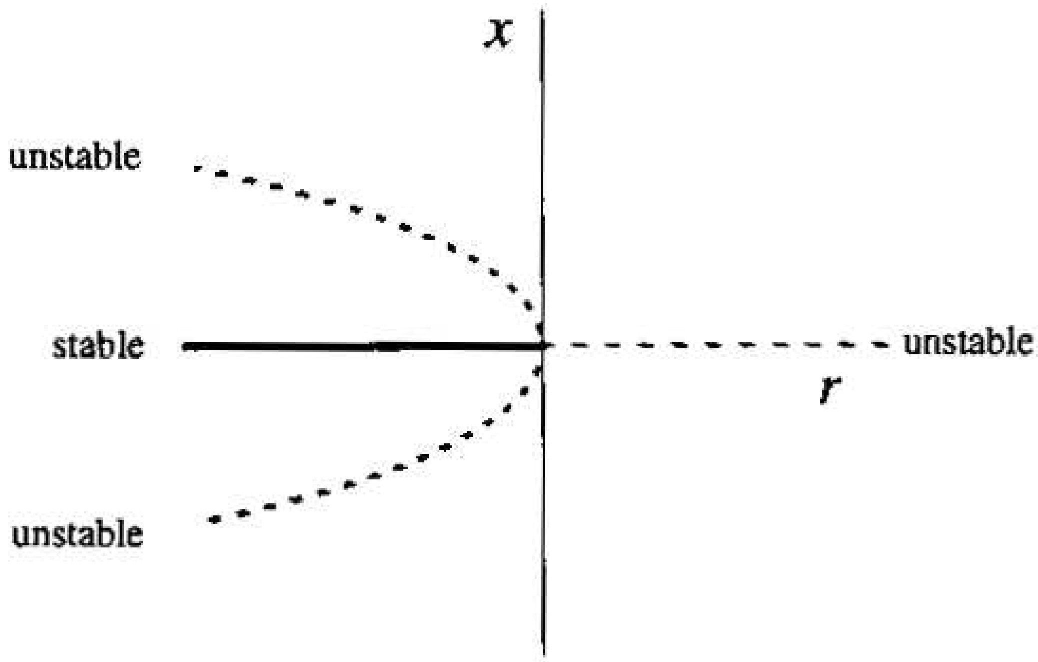


Fig. 3.4.5

- Non-zero fixed points are unstable and exist only below the bifurcation ($r < 0$)

- cf Fig. 3.4.2

In real physical systems, higher order terms are likely to be present....e.g. $\dot{x} = rx + x^3 - x^5$

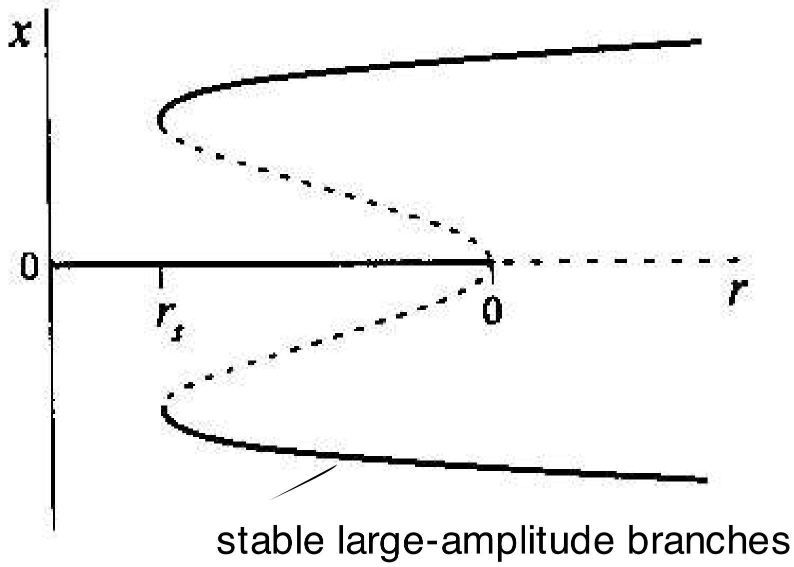


Fig. 3.4.6

- In the range $r_s < r < 0$, two *qualitatively different* stable states *coexist*.

- The initial condition x_0 determines which fixed point is approached as $t \to \infty$.

- Origin is *locally*, but *not globally*, stable

The existence of different stable states allows for the possibility of *jumps* and *hysteresis* [associated with "memory" of system] as r is varied.

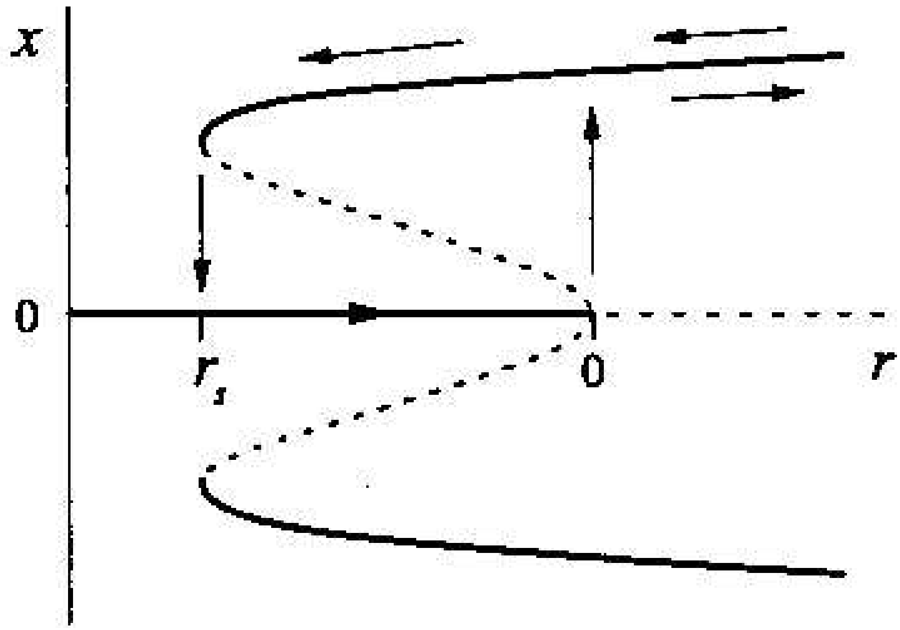


Fig. 3.4.7

Bifurcation at r_s is a saddle-point bifurcation in which stable and unstable fixed points are born "out of the clear blue sky" as r is increased.

3.4 More on terminology...

- The “Supercritical pitchfork” is sometimes called a “forward bifurcation” and is closely related to a continuous or second-order phase transition in statistical mechanics.

- The “Subcritical pitchfork” is sometimes called an “inverted” or “backward” bifurcation, and is related to discontinuous or first-order phase transitions.

- In engineering, “supercritical” bifurcations are sometimes called “soft” or “safe”, while “subcritical” are “hard” or “dangerous”!

3.6 Insect Outbreaks and Catastrophes

Ludwig (1978) proposed the following model for the budworm population dynamics

$$\dot{N} = RN\left[1 - \frac{N}{K}\right] - p(N) \qquad (1)$$

$$p(N) = \frac{BN^2}{A^2 + N^2} \qquad (2)$$

where A, B are constants (> 0), $N(t)$ is the budworm population, R is the growth rate, K is carrying capacity (recall section 2.3) and $p(N)$ represents the death rate due to predators (e.g. birds).

Recast the equation in dimensionless form...

$$\frac{dx}{d\tau} = rx\left[1 - \frac{x}{k}\right] - \frac{x^2}{1 + x^2}, \qquad (3)$$

where $x = N/A$, $\tau = Bt/A$, $r = RA/B$, $k = K/A$.

- There is a fixed point at $x^* = 0$, by inspection, which is *always unstable* $\Rightarrow$ predation is weak for small x.

- Hence the budworm population grows exponentially for x near zero

- Other fixed points obtained from $dx/d\tau = 0$ so that

$$r\left[1 - \frac{x}{k}\right] = \frac{x}{1 + x^2}$$

Let's plot these two curves and study their intersections as k and r vary.....

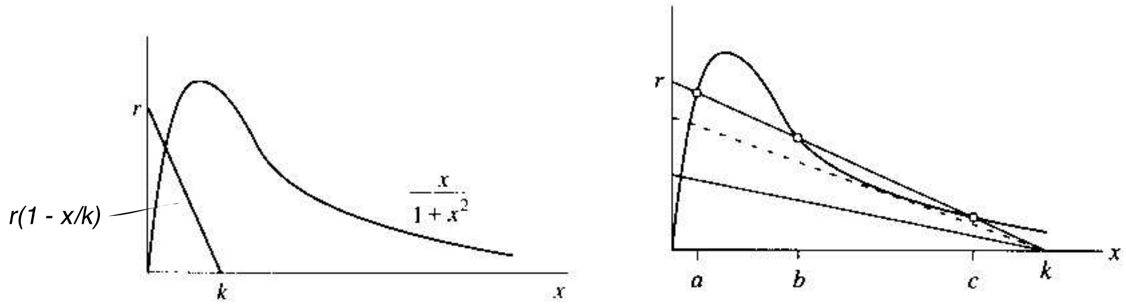


Fig. 3.6.1

- Small $k \Rightarrow$ one intersection for any $r > 0$

- Large $k \Rightarrow$ 1, 2 or 3 intersections, depending on r

- b and c coalesce in a saddle-node bifurcation as r decreases (see dashed line)

- a and b coalesce as r increases

Stability of fixed points

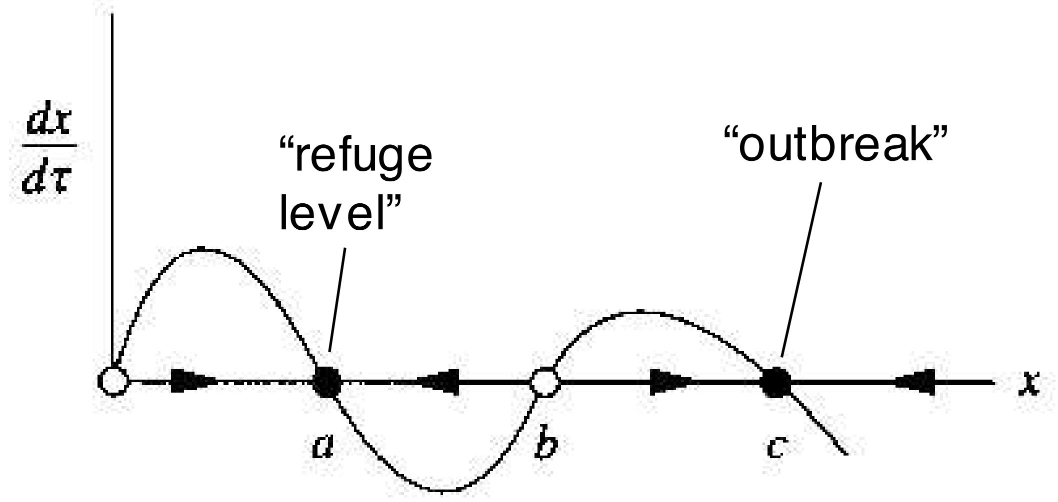


Fig. 3.6.2

- Vector field shown for a regime where there exist 3 fixed points in addition to $x^* = 0$ (unstable)

- Fate of the system determined by the initial condition x_0

- $\Rightarrow$ outbreak occurs if and only if $x_0 > b$, hence b represent a *threshold*.

An outbreak can also be triggered by a saddle-node bifurcation: if r and k drift in such a way that $x^* = a$ disappears, then the population will jump suddenly to outbreak level c. The situation is made worse by the *hysteresis effect* - even if the parameters are restored to their values before the outbreak, the population will not drop back to the refuge level.

Calculating the Bifurcation Curves

We now discuss the bifurcation curves in (k, r) space where the system undergoes saddle-node bifurcations.

The condition for a saddle-node bifurcation is that $r[1 - x/k]$ intersects $x/(1 + x^2)$ tangentially (recall Example 3.1.2). Hence

$$r\left[1 - \frac{x}{k}\right] = \left[\frac{x}{1 + x^2}\right] \qquad (4)$$

and

$$\frac{d}{dx}\left(r\left[1 - \frac{x}{k}\right]\right) = \frac{d}{dx}\left[\frac{x}{1 + x^2}\right] \qquad (5)$$

Evaluate (5) and eliminate k from (4) to yield

$$r = \frac{2x^3}{(1+x^2)^2} \tag{6}$$

Insert (6) into (5) to yield

$$k = \frac{2x^3}{x^2-1} \tag{7}$$

Since $k > 0$ hence $x > 1$. Equations (6) an (7) define the **bifurcation curves**.

$$r = \frac{2x^3}{(1+x^2)^2} \equiv r(x) \tag{8}$$

$$k = \frac{2x^3}{x^2-1} \equiv k(x) \tag{9}$$

To generate *bifurcation curves* in the (k, r) plane:

(i) Choose x where $x > 1$

(ii) Plot the point $(k(x), r(x))$ in the (k, r) plane

(iii) Repeat (i) and (ii) for all $x > 1$.

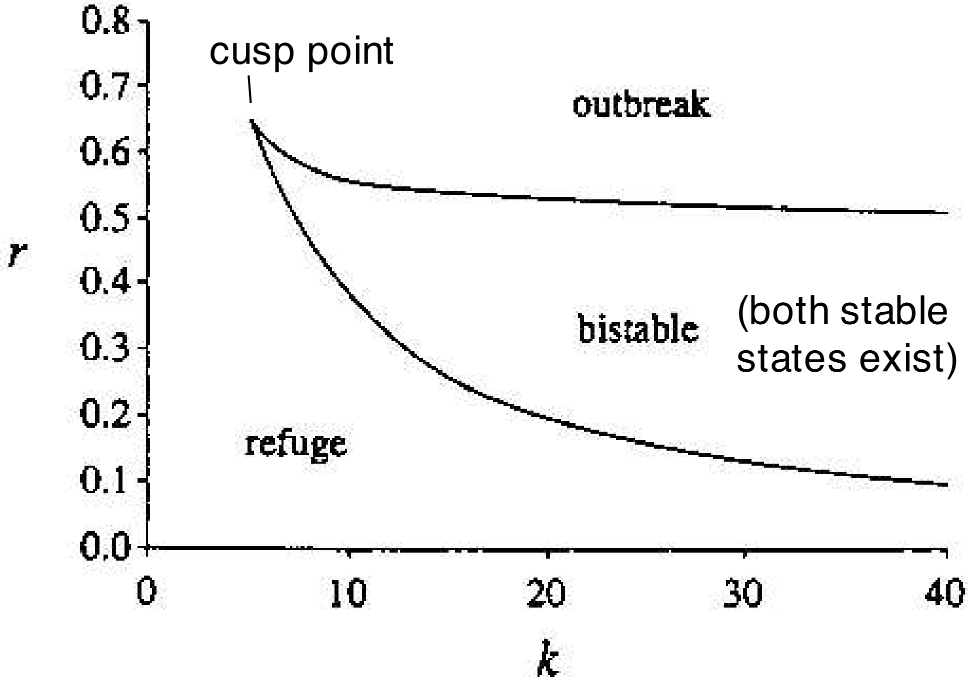


Fig. 3.6.3

We could also plot the fixed points x^* above the (k, r) plane to yield a 3-dimensional surface called a <u>cusp catastrophe</u> surface...

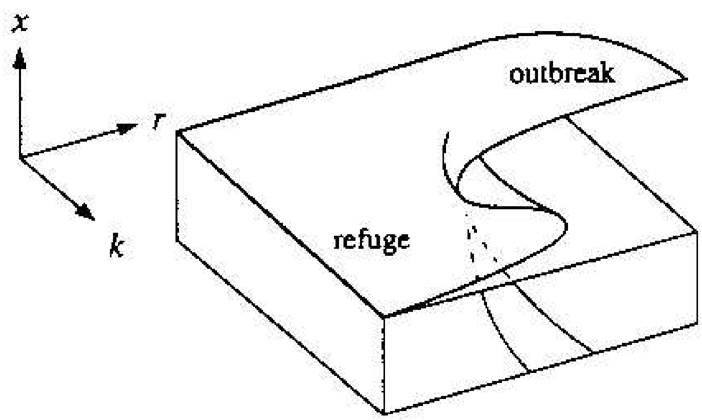


Fig. 3.6.4

- The *cusp catastrophe surface* folds over on itself in certain places.

- The *projection* of these folds onto the (k, r) plane yields the *bifurcation curves* shown above.

- The term "catastrophe" is motivated by the fact that, as parameters change, the state of the system can be carried over the edge of the upper surface, after which it drops discontinuously to the lower surface.

- This jump could be truly catastrophic for the equilibrium e.g. of a bridge or a building!