

# Physics of Atmospheres and Oceans: Class Question Sheets

## RADIATION

\* Indicates optional questions or parts of questions

R.1 The spectral radiance  $L(\tilde{\nu})$  (as a function of wavenumber  $\tilde{\nu}$ ) emitted by the surface of a black body at temperature  $T$  is the Planck spectral radiance  $B_{\tilde{\nu}}(\tilde{\nu})$ , given by

$$L(\tilde{\nu}) = B_{\tilde{\nu}}(\tilde{\nu}) = \frac{2hc^2\tilde{\nu}^3}{\exp(hc\tilde{\nu}/kT) - 1}$$

(i) Find the expression for  $B_{\lambda}(\lambda)$ , the Planck spectral radiance as a function of wavelength.

(ii) Clearly, when plotted on a linear wavenumber scale, equal areas under  $B_{\tilde{\nu}}(\tilde{\nu})$  correspond to equal radiances; for what function is this true if a log scale is chosen for  $\tilde{\nu}$ ?

(iii) Show that the values of  $\tilde{\nu}$  and  $\lambda$  that make  $B_{\tilde{\nu}}(\tilde{\nu})$  and  $B_{\lambda}(\lambda)$  maximum are given by

$$\frac{\tilde{\nu}_{\max}}{T} = c_1 \quad \text{and} \quad \lambda_{\max}T = c_2,$$

i.e.  $\tilde{\nu}_{\max}$  and  $\lambda_{\max}$  do not correspond to the same photon energy. Given that  $c_2 = 0.29$  cm K, find the temperatures for which  $B_{\lambda}$  is maximum at (a) 500 nm (b) 10  $\mu\text{m}$ .

(iv) Assuming that the sun behaves as a black body at a temperature at a temperature of 5800 K, calculate  $B_{\lambda}(\lambda)$  and  $B_{\tilde{\nu}}(\tilde{\nu})$  at 500 nm, and the percentage increase in these spectral radiances if the temperature increases by 100 K.

[ $2hc^2 = 1.1911 \times 10^{-8} \text{ Wm}^{-2}\text{sr}^{-1}(\text{cm}^{-1})^{-4}$  and  $hc/k = 1.439 \text{ K}(\text{cm}^{-1})^{-1}$ . Unfortunately,  $\tilde{\nu}$  is still conventionally measured in ‘wavenumbers’, i.e. inverse centimetres.]

R.2 (i) Elements of surface  $\delta S_1$  and  $\delta S_2$  are at a distance  $r$  apart, and with normals inclined at angles  $\theta_1$  and  $\theta_2$  to the line joining them. Under what conditions is the net radiative power flow between them in optical passband  $\delta\tilde{\nu}$  given by

$$\delta P_{1\rightarrow 2} = \frac{B_{\tilde{\nu}}(\tilde{\nu}, T_1)}{r^2} \delta\tilde{\nu} \delta S_1 \delta S_2 \cos\theta_1 \cos\theta_2 ?$$

(ii) Integrate  $B_{\tilde{\nu}}$  over  $\tilde{\nu}$  and hence calculate the solar constant  $E_s$  (i.e. the normal irradiance just outside the atmosphere) assuming that the sun is a black body at 5800 K.  $\left[ \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \right]$ . The radius of the sun is  $7 \times 10^8$  m and the radius of the Earth's orbit is  $1.5 \times 10^{11}$  m. ]

(iii)\* Carry out the integration over the whole hemisphere which can be seen by  $\delta S_1$ , find the expression for Stefan's constant, and calculate its value.

(iv)\* By carrying out the appropriate integration, find the power absorbed by a horizontal black disk of radius  $a$  from an infinite horizontal black plane at temperature  $T_1$  over which it is suspended at height  $h$ .

(v) Find the answer to (iv) using symmetry and thermodynamic arguments, and any results you wish from (ii) and (iii). Show also the power absorbed from the plane by a black sphere of radius  $a$  is  $2\pi a^2 \sigma T_1^4$ .

R.3 A low altitude earth satellite in an equatorial orbit carries below it two isolated spherical radiation sensors of negligible heat capacity. One is painted white, and may be assumed to be perfectly reflecting at all wavelengths at which there is significant solar energy ( $\lambda < 4 \mu\text{m}$ ) and perfectly absorbing at longer wavelengths. The other is painted black and is perfectly absorbing at all wavelengths. Assume there is no direct or scattered energy input from the spacecraft.

(i) Calculate the temperatures of the two spheres at midnight over a thick cloud sheet at a temperature of 280 K. (Use the result from the end of R.2(v).)

(ii) Find the radiance due to diffusely scattered sunlight just above a cloud of albedo  $\alpha$  when the sun is overhead.

(iii) Assuming the spheres are shadowed from overhead direct sunlight by the spacecraft, calculate their temperatures over a thick cloud of temperature 280 K and albedo 0.8, given the solar constant  $E_s = 1370 \text{ W m}^{-2}$ .

R.4 Derive the radiative transfer equation

$$\frac{dL_{\tilde{\nu}}}{dz} = \frac{k_{\tilde{\nu}} \rho_a}{\cos\theta} (B_{\tilde{\nu}} - L_{\tilde{\nu}})$$

for the spectral radiance  $L_{\tilde{\nu}}(z, \theta)$  travelling upwards at an angle  $\theta$  to the vertical in a planetary atmosphere in local thermodynamic equilibrium, where  $\rho_a$  is the absorber density, and  $k_{\tilde{\nu}}$  is the absorption coefficient.

R.5 In the ‘two-stream’ approximation, the radiative transfer equation is integrated over upward and downward facing hemispheres, and the following one-dimensional equations are obtained:

$$-\frac{dE_{\tilde{\nu}}^{\uparrow}}{d\chi_{\tilde{\nu}}^*} = \pi B_{\tilde{\nu}} - E_{\tilde{\nu}}^{\uparrow}$$

$$\frac{dE_{\tilde{\nu}}^{\downarrow}}{d\chi_{\tilde{\nu}}^*} = \pi B_{\tilde{\nu}} - E_{\tilde{\nu}}^{\downarrow}$$

where  $\chi_{\tilde{\nu}}^* = 1.66 \int_z^{\infty} k_{\tilde{\nu}} \rho_a dz$ , a positive quantity, is the optical path from  $z$  to  $\infty$ ,  $\rho_a$  is the absorber density, and  $k_{\tilde{\nu}}$  is the absorption coefficient,  $E_{\tilde{\nu}}^{\uparrow}$  and  $E_{\tilde{\nu}}^{\downarrow}$  are the upward and downward spectral irradiances, and  $B_{\tilde{\nu}}(T)$  is the Planck function.

For an atmosphere containing an absorber with a constant mass mixing ratio  $x$  and an absorption coefficient  $k$  that is independent of  $z$ , find an expression for  $\chi_{\tilde{\nu}}^*$  as a function of the pressure  $p$  at level  $z$ . For the case where  $\chi^*$  is independent of  $\tilde{\nu}$ , write down the corresponding equations for the spectrally-integrated irradiances  $E^{\uparrow}$ ,  $E^{\downarrow}$  and Planck function  $B(T)$ .

R.6 Consider an atmosphere that is transparent to solar radiation, with the ground at temperature  $T_g$ . The only absorption within the atmosphere takes place in the infra-red. The atmosphere contains a single absorbing gas, of constant mixing ratio  $x$  and absorption coefficient  $k$ . Assume that the atmosphere and surface are in radiative equilibrium and that the spectrally-integrated irradiances  $E^{\uparrow}$  and  $E^{\downarrow}$  are continuous between the ground and the atmosphere just above the ground.

(i) Show that the net flux  $\phi = E^{\uparrow} - E^{\downarrow} = \text{constant}$ .

(ii) Show that

$$\pi B(T) = \frac{1}{2} (E^{\uparrow} + E^{\downarrow})$$

and hence that

$$2\pi B(T) = \left( \frac{1.66kxp}{g(1+x)} + 1 \right) \phi.$$

(iii) Show that  $\phi = 2\pi[B(T_g) - B(T_b)]$ , where  $T_b$  is the temperature at the bottom of the atmosphere. (Note that in this model,  $T_b \neq T_g$  in general.)

(iv) Find  $\phi$ , given that at 250 mb the temperature is 220 K, and that  $T_g = 280$  K. [Hint: you will need  $\chi^*(250 \text{ mb})/\chi^*(1000 \text{ mb})$ .]

(v) Find the temperature discontinuity  $T_g - T_b$  at the surface. What does your answer imply about radiative equilibrium?

(vi) Plot  $E^{\uparrow}$ ,  $E^{\downarrow}$  and  $\pi B$  as functions of  $p$ .

R.7\* *Outline* (i.e. don't copy the full details from a textbook) the Lorentz description of the pressure broadening of spectral lines leading to the expression for the absorption coefficient

$$k_{\tilde{\nu}} = \frac{S\gamma_L}{\pi[(\tilde{\nu} - \tilde{\nu}_0)^2 + \gamma_L^2]}$$

where

$$\gamma_L = \gamma_{L0} \left(\frac{p}{p_0}\right) \left(\frac{T_0}{T}\right)^{\frac{1}{2}}$$

What are the main shortcomings of this description? Write down the conditions under which the absorption  $(1 - \tau_{\tilde{\nu}})$  by a uniform path of total pressure  $p$ , length  $l$ , and absorber density  $\rho_a$  due to a pressure broadened line is (i) strong ( $\tau_{\tilde{\nu}_0} \ll 1$ ) and (ii) weak ( $1 - \tau_{\tilde{\nu}_0} \ll 1$ ) and derive expressions for the equivalent width in both cases.

Under what conditions is the transmittance at  $\tilde{\nu}_0$  for such a path independent of the pressure?

R.8\* *Outline* a description of Doppler broadening leading to the expression

$$k_{\tilde{\nu}} = \frac{S}{\gamma_D \pi^{\frac{1}{2}}} \exp \left[ - \left( \frac{\tilde{\nu} - \tilde{\nu}_0}{\gamma_D} \right)^2 \right]$$

where  $\gamma_D = (\tilde{\nu}_0/c)(2RT/M)^{\frac{1}{2}}$  and  $M$  is the mass of one mole. Derive expressions for the strong\* and weak equivalent widths as in the previous question.

R.9 Explain the Curtis-Godson approximation and show that the spectral mean of atmospheric transmittances calculated using it are accurate when (i) strong pressure broadening conditions exist or (ii) the absorption is weak, irrespective of the line shape.

Derive the Curtis-Godson pressure for an atmospheric absorber with constant mass mixing ratio  $x$  for a vertical path from height  $z$  to  $\infty$ .

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