7. Flows in a rotating reference frame

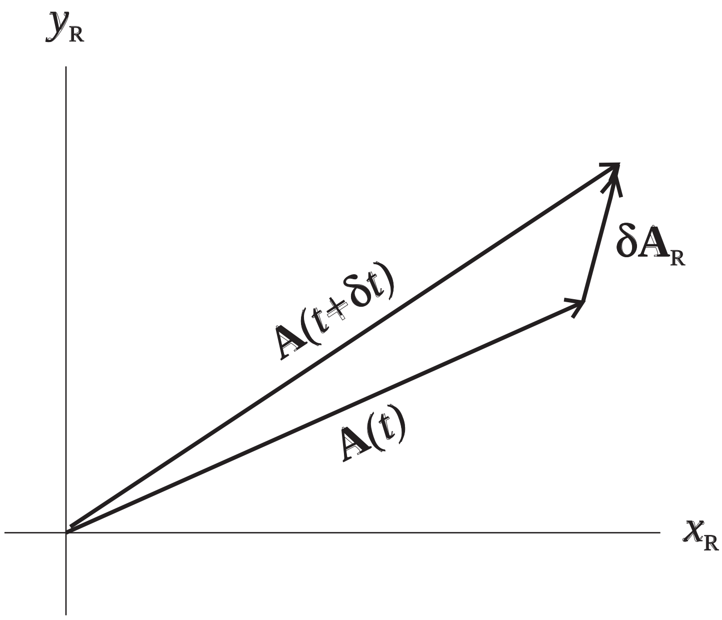
7.1. Rotating reference frames

When fluids move within a reference frame that is rapidly rotating, they are subject to new (gyroscopic) constraints that can significantly affect the behaviour of the flow. In particular, this can introduce an anisotropic "stiffness" that can render the flow almost two-dimensional under some circumstances. This is of particular relevance in some engineering contexts (turbomachinery etc.) and also for atmospheres and oceans (for which the planetary rotation may be important).

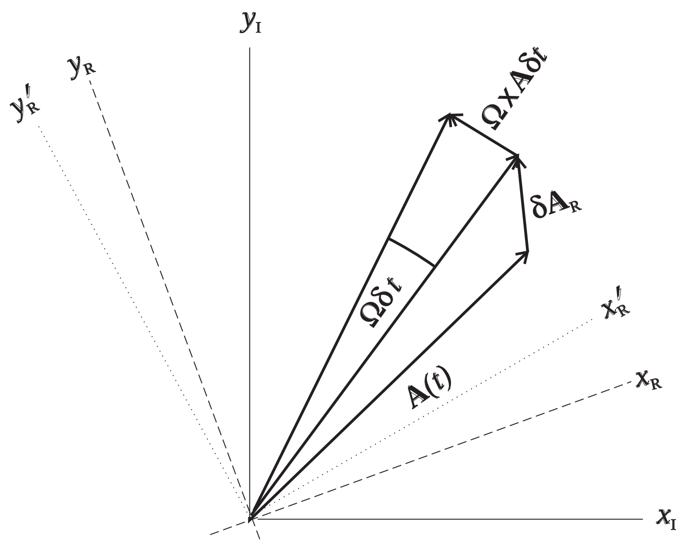
We first consider how the equations of motion are modified in a rotating reference frame. We use rotating axes, fixed w.r.t. the rotating system but note that time-derivatives of vectors must be modified:

Suppose frame R rotates at constant angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$ w.r.t. an inertial frame I.

Consider vector $\boldsymbol{A}(t)$ – suppose for simplicity that it lies in the $xy-$plane of R and I. As viewed in R, in time interval δt, $\boldsymbol{A} \to \boldsymbol{A}(t + \delta t) \approx \boldsymbol{A}(t) + \delta \boldsymbol{A}_{\mathrm{R}}$, where $\delta \boldsymbol{A}_{\mathrm{R}} = \left(\frac{d\boldsymbol{A}}{dt}\right)\delta t$:



But if the same $\boldsymbol{A}$ is viewed in I, we must allow for the rotation of frame R w.r.t frame I in time δt; this gives an extra bit of $\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{I}}$:



The extra bit $= \boldsymbol{\Omega} \times \boldsymbol{A}\delta t$. (Note that the same works if $\boldsymbol{A}$ is not in the $xy-$plane.) So as $\delta t \to 0$,

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{A} \tag{7.1}$$

Repeat this procedure to obtain the 2nd time derivative:

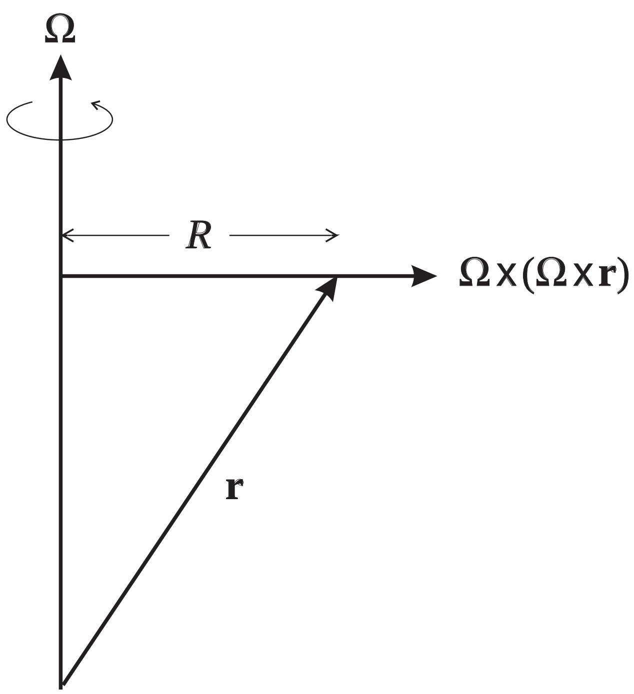
$$\begin{aligned}
\left(\frac{d^2\boldsymbol{A}}{dt^2}\right)_{\mathrm{I}} &= \left(\frac{d}{dt}\right)_{\mathrm{I}} \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{I}} \\
&= \left(\frac{d}{dt}\right)_{\mathrm{I}} \left(\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{A}\right) \\
&= \left(\frac{d}{dt}\right)_{\mathrm{R}} \left(\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{A}\right) \\
&\quad + \boldsymbol{\Omega} \times \left[\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{A}\right] \\
&= \left(\frac{d^2\boldsymbol{A}}{dt^2}\right)_{\mathrm{R}} + 2\boldsymbol{\Omega} \times \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{A})
\end{aligned}$$

In particular, if $\boldsymbol{A} = \boldsymbol{r}$, the position vector, we get the velocity and acceleration:

$$\boldsymbol{a}_{\mathrm{I}} = \boldsymbol{a}_{\mathrm{R}} + 2\boldsymbol{\Omega} \times \boldsymbol{u}_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \qquad (7.2)$$

• $2\boldsymbol{\Omega} \times \boldsymbol{u}_{\mathrm{R}}$ is the *Coriolis acceleration*: it is perpendicular to the velocity $\boldsymbol{u}_{\mathrm{R}}$ in the rotating frame, and to $\boldsymbol{\Omega}$.

• $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ is the *centripetal acceleration*: it has magnitude $\Omega^2 R$, and points perpendicularly away from the rotation axis:



• Now replace $\boldsymbol{a}$ in Newton 2 by $\boldsymbol{a}_{\mathrm{I}}$, drop subscript R, to get the equation of motion *with respect to the rotating frame:*

$$\boxed{\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}\times\boldsymbol{u} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) - g\boldsymbol{k} + \nu\nabla^2\mathbf{u}} \tag{7.3}$$

where $\boldsymbol{u}$ is now the velocity measured in the rotating frame R.

• We often put $\boldsymbol{g}' = -g\boldsymbol{k} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$, the *effective gravity*. This is because these terms can be combined and represented as $-\nabla\Phi$, where $\Phi = gz + \frac{1}{2}\Omega^2 R^2$ is an 'effective geopotential' function.

This allows us to simplify the N-S equation to

$$\boxed{\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}\times\boldsymbol{u} - \nabla\Phi + \nu\nabla^2\mathbf{u}} \tag{7.4}$$

7.2. *Geostrophic flow*

We see, then, that the principal difference that rotation is likely to make to the flow of a homogeneous fluid is in the action of the Coriolis acceleration. An important class of flows is obtained when motion is dominated by Coriolis accelerations.

Consider a steady flow in which the Coriolis effect is large compared with both the inertia of the relative motion and viscous forces. This means that

$$|\boldsymbol{u}.\nabla\boldsymbol{u}| \ll |\boldsymbol{\Omega}\times\boldsymbol{u}| \quad \text{and} \quad |\nu\nabla^2\boldsymbol{u}| \ll |\boldsymbol{\Omega}\times\boldsymbol{u}|. \tag{7.5}$$

The ratios of these terms can be expressed in terms of dimensionless numbers, measured with respect to typical length and velocity scales, L and U, as

$$\frac{|\boldsymbol{u}.\nabla\boldsymbol{u}|}{|\boldsymbol{\Omega}\times\boldsymbol{u}|} \sim \frac{U}{\Omega L} = \mathrm{Ro} \tag{7.6}$$

and

$$\frac{|\nu\nabla^2\boldsymbol{u}|}{|\boldsymbol{\Omega}\times\boldsymbol{u}|} \sim \frac{\nu}{\Omega L^2} = \mathrm{Ek}, \tag{7.7}$$

where Ro is the Rossby number and Ek is the Ekman number respectively.

As in a non-rotating fluid, viscous effects may become important, even when simple order of magnitude considerations suggest otherwise, in boundary layers close to surfaces. Under conditions when Ro $\ll 1$ and Ek $\ll 1$, however, well away from any boundaries, and for which the 'effective gravity' is negligible, then the N-S equation (7.4) simplifies to

$$\boxed{2\boldsymbol{\Omega} \times \boldsymbol{u} \simeq -\frac{1}{\rho}\nabla p.} \tag{7.8}$$

Flows in which this balance of forces between Coriolis effects and pressure gradient forces is dominant are called geostrophic flows.

• Because the Coriolis 'force' is always directed perpendicular to the flow direction, a consequence of geostrophic balance is that the pressure gradient is also at right angles to the flow.

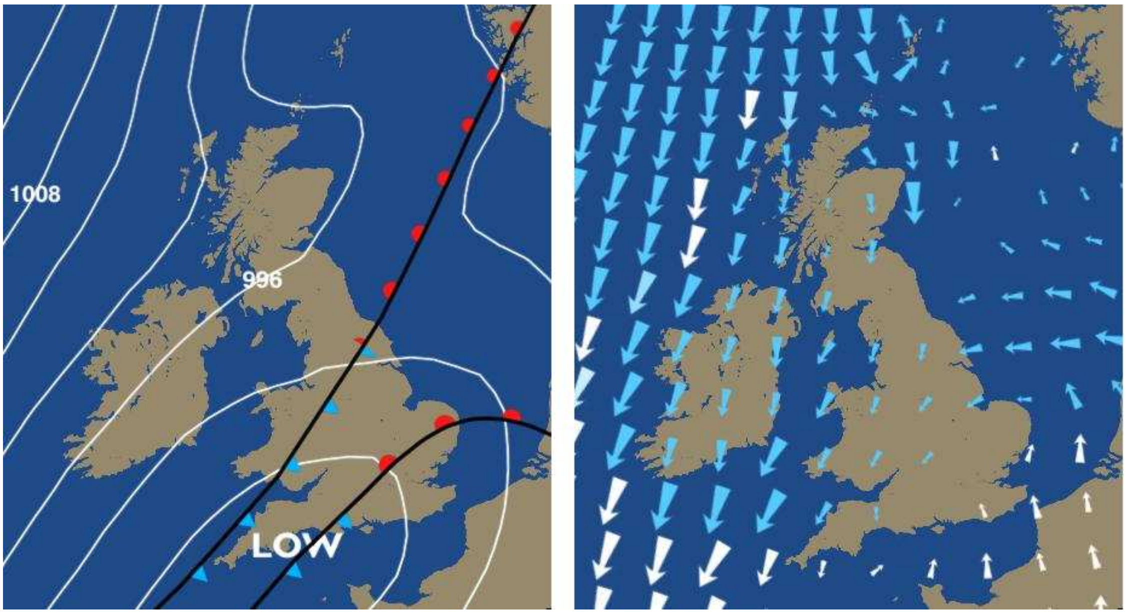
Mathematically, consider the case where $\mathbf{\Omega} = \Omega\mathbf{k}$ and $\mathbf{u} = (u, v, 0)$. Taking $\mathbf{k}\times$ Eq (7.8)

$$\mathbf{k} \times (2\mathbf{\Omega} \times \mathbf{u}) = -2\Omega\mathbf{u} \tag{7.9}$$

so that for an incompressible, homogeneous fluid

$$\mathbf{u} = \mathbf{k} \times \nabla \left(\frac{p}{2\Omega\rho}\right). \tag{7.10}$$

Thus, $p/(2\Omega\rho)$ acts like a *streamfunction* for geostrophically balanced horizontal flow (cf section 4.2).



A commonplace application of this can be seen in weather maps, that typically show contours of surface pressure variations (see above left, taken from the BBC weather website for 4 February 2009). The corresponding wind vectors (above right) are more or less parallel to the pressure contours....

7.3. Taylor-Proudman Theorem

Another interesting consequence of strong background rotation can be found by taking the curl of Eq (7.8). For a homogeneous fluid, the curl of the RHS of Eq (7.8) is identically zero, leaving the result

$$\nabla \times (2\mathbf{\Omega} \times \mathbf{u}) \simeq 0. \qquad (7.11)$$

This expression expands into

$$(2\mathbf{\Omega}.\nabla)\mathbf{u} - (\mathbf{u}.\nabla)\mathbf{2\Omega} + \mathbf{u}(\nabla.(2\mathbf{\Omega})) - 2\mathbf{\Omega}(\nabla.\mathbf{u}) = 0. \qquad (7.12)$$

But $\mathbf{\Omega}$ is not a function of position and $\nabla.\mathbf{u} = 0$ for incompressible flow. So we are left with

$$(2\mathbf{\Omega}.\nabla)\mathbf{u} = 0. \qquad (7.13)$$

Where $\mathbf{\Omega} = (0, 0, \Omega)$, this implies

$$\frac{\partial \mathbf{u}}{\partial z} = 0 \qquad (7.14)$$

which essentially means that there is no variation of the velocity field in the direction parallel to the axis of rotation.

• This result is known as the Proudman-Taylor Theorem after the oceanographer Joseph Proudman, who first published this theoretical result in 1915 and G. I. Taylor, who demonstrated it experimentally.

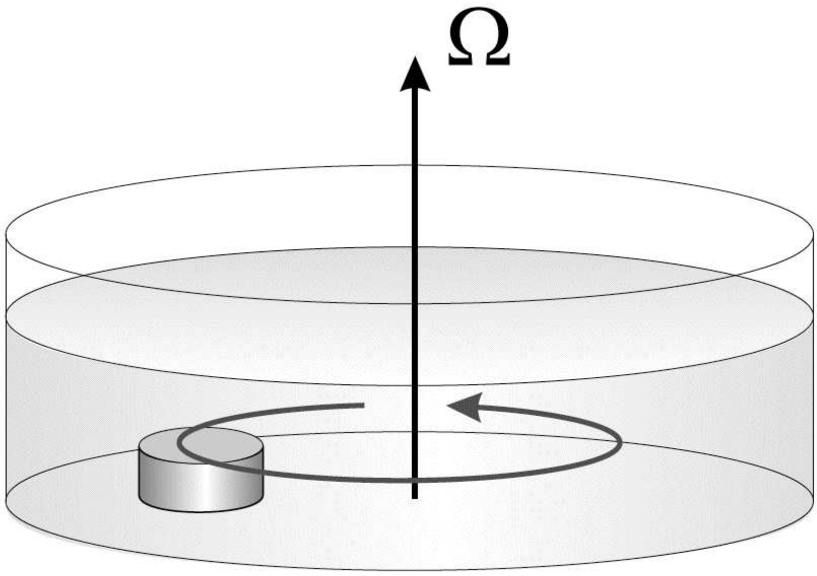
• For a system with solid boundaries perpendicular to the rotation axis (so $w = 0$), then Eq (7.14) implies that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \qquad (7.15)$$

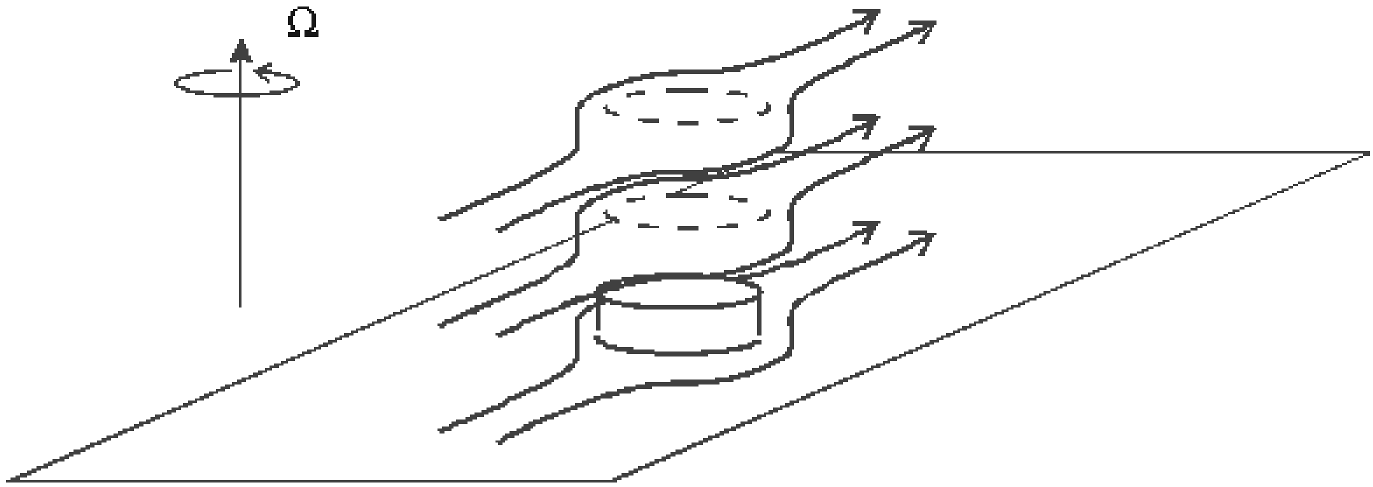
everywhere!

• Thus the Proudman-Taylor theorem says that flow will be almost entirely two-dimensional in planes perpendicular to the axis of rotation.

One way in which this can be shown experimentally is to set up a slow azimuthal flow in a rapidly-rotating tank of fluid with a shallow obstacle in the bottom, as in the figure below.



The slow flow is produced by making a small change to the rotation of the tank after the fluid has been brought into solid-body rotation by spinning the apparatus for some time beforehand.



• As the flow encounters an obstacle on the bottom, the whole flow moves around the obstacle instead of flowing over the top, and behaves as if the obstacle extended throughout the depth of the tank. This is shown schematically above (adapted from http://www-paoc.mit.edu/labweb/experiments.htm).

• The resulting almost two-dimensional flow around the 'virtual obstacle' associated with the shallow bump is sometimes called a Taylor column after G. I. Taylor, who first observed it.

7.4. Ekman Layers

The geostrophic flows introduced above will not generally satisfy the non-slip (or fixed stress) boundary conditions applicable at the edges of a rotating container. This means that another force must come into play to enable the geostrophic flow in the main body of the fluid to match the boundary conditions - and that additional force is most likely to be the viscous force. This depends on a high order derivative of the velocity field, so will lead to the formation of a thin boundary layer - the Ekman layer - after the Swedish oceanographer Vagn Ekman who discovered it in the early 1900s.

Consider a body of fluid in a container of depth H, rotating at angular velocity Ω about the z-axis with a solid boundary in the xy-plane. There is a uniform, steady geostrophic flow $(u_0, 0, 0)$ in the main body of the fluid, which satisfies the usual balance conditions with respect to the pressure p_0

$$0 = -\frac{1}{\rho}\frac{\partial p_0}{\partial x}, \tag{7.16}$$

$$2\Omega u_0 = -\frac{1}{\rho}\frac{\partial p_0}{\partial y}. \tag{7.17}$$

We now suppose that the viscous force comes into play in a narrow layer of depth $\sim h \ll H$, such that $\partial/\partial z \gg \partial/\partial x$ or $\partial/\partial y$ within this layer. In this case the steady state Navier-Stokes and continuity equations become

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \tag{7.18}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}, \tag{7.19}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2}, \tag{7.20}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{7.21}$$

The last equation above implies that $w \ll u$ or v, so that $\partial p/\partial z \simeq 0$ and p is a function of x and y only.

In the presence of the geostrophic flow above, this means that

$$\frac{\partial p}{\partial x} = \frac{\partial p_0}{\partial x} = 0; \quad \frac{\partial p}{\partial y} = \frac{\partial p_0}{\partial y}. \tag{7.22}$$

Substituting into the horizontal momentum equations and making use of Eq (7.17) we obtain

$$-2\Omega v = \nu \frac{\partial^2 u}{\partial z^2}, \tag{7.23}$$

$$-2\Omega(u_0 - u) = \nu \frac{\partial^2 v}{\partial z^2}, \tag{7.24}$$

with boundary conditions $u = v = 0$ at $z = 0$ and $u \to u_0$ and $v \to 0$ as $z \to \infty$.

This can be solved in principle by eliminating u or v to give a fourth order differential equation. However, a simpler approach is to introduce the complex variable

$$Z = (u + iv). \tag{7.25}$$

Taking $i\times$ Eq (7.24) + Eq (7.23) leads to the 2nd order equation

$$2i\Omega(Z - u_0) = \nu \frac{\partial^2 Z}{\partial z^2} \tag{7.26}$$

with boundary conditions $Z \to 0$ as $z \to 0$ and $Z \to u_0$ as $z \to \infty$. This has the solution

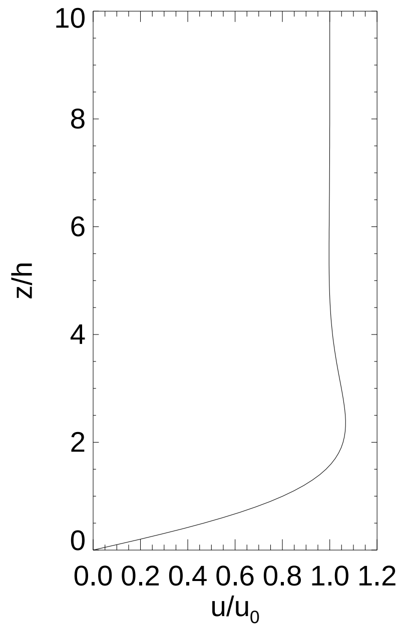
$$Z = u_0 \left(1 - \exp\left[\left(\frac{2i\Omega}{\nu} \right)^{1/2} z \right] \right) \tag{7.27}$$

or, taking real and imaginary parts,

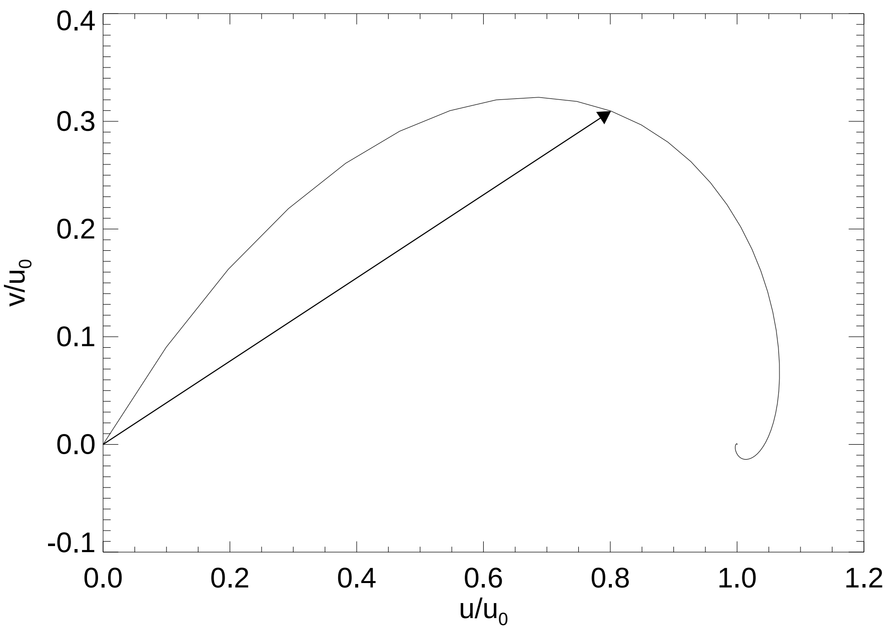
$$\begin{aligned} u &= u_0 \left(1 - e^{-z/h} \cos\left[\frac{z}{h}\right] \right) && (7.28) \\ v &= u_0 e^{-z/h} \sin\left[\frac{z}{h}\right], && (7.29) \end{aligned}$$

where $h = (\nu/\Omega)^{1/2}$ is the 'depth' of the boundary layer.

• The velocity components of this solution exhibit damped oscillations with height z towards $(u_0, 0)$, as required by the boundary conditions.



- Seen from above, the velocity vector forms a spiral profile, known as the Ekman spiral



- The arrow shows the velocity vector at $z = h$.

With a more general geostrophic flow in the body of the fluid $\mathbf{u} = (u_0, v_0)$ (independent of z but possibly dependent on x and y), the flow in the Ekman layer would be

$$
\begin{aligned}
u &= u_0 \left(1 - e^{-z/h} \cos\left[\frac{z}{h}\right]\right) - v_0 e^{-z/h} \sin\left[\frac{z}{h}\right] \quad (7.30) \\
v &= v_0 \left(1 - e^{-z/h} \cos\left[\frac{z}{h}\right]\right) + u_0 e^{-z/h} \sin\left[\frac{z}{h}\right] \quad (7.31)
\end{aligned}
$$

Consider $\partial/\partial x$ Eq (7.30) + $\partial/\partial y$ Eq (7.31):

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= \left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) \left(1 - e^{-z/h} \cos\left[\frac{z}{h}\right]\right) - \\
&\quad \left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right) e^{-z/h} \sin\left[\frac{z}{h}\right] \\
&= -\frac{\partial w}{\partial z}.
\end{aligned}
$$

Note, however, that $\partial u_0/\partial x + \partial v_0/\partial y = 0$ from the definition of geostrophic flow (*check* from Eqs (7.8) and (7.10)!). Thus, integrating $\partial w/\partial z$ from $z = 0$ to $z = \infty$, noting that $w = 0$ at $z = 0$, we obtain the vertical velocity w_E at the top of the Ekman layer

$$w_E = \frac{h}{2}\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right) = \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{1/2} \omega_0. \qquad (7.32)$$

• This means that the Ekman layer leads to vertical motion $w \neq 0$ in the geostrophic interior if the vorticity of the geostrophic flow $\omega_0 \neq 0$. This vertical velocity is known as the Ekman suction velocity.

• If the lower boundary is made to rotate at a different angular velocity Ω_B from the geostrophic flow, this produces an Ekman vertical velocity that will lead to a secondary ageostrophic circulation that helps the fluid to change its rotation towards Ω_B, a process known as spin-up or spin-down (see Acheson Chapter 8.5).

• This adjustment of the geostrophic flow takes place on a timescale

$$\tau = \frac{1}{2\Omega}\frac{H}{h} = \frac{H}{2(\Omega\nu)^{1/2}}, \qquad (7.33)$$

which may be much shorter than the diffusive timescale H^2/ν.