Paper B3: Physics of Fluid Flows

Problem Set

1. Consider a steady two-dimensional flow rotating with constant angular velocity Ω. In Cartesian coordinates the velocity is $\mathbf{u} = (u, v, w) = (-\Omega y, \Omega x, 0)$. (This is called *solid body rotation.*)

Calculate the Cartesian components of $(\mathbf{u} \cdot \nabla)\mathbf{u}$ for this flow, and verify that it represents the familiar centripetal acceleration $\Omega^2 r$ towards the rotation axis. Write down $\dfrac{\partial \mathbf{u}}{\partial t}$ and $\dfrac{D\mathbf{u}}{Dt}$, and show that they are not equal for this flow. [Take care to hold the correct variables constant while calculating partial derivatives.] Give a physical explanation of why they are different.

2. Starting from the continuity equation in Eulerian form, $\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{u}) = 0$, derive the Lagrangian form,

$$D\rho/Dt + \rho\nabla \cdot \mathbf{u} = 0\ .$$

Give a physical interpretation of the latter equation in terms of the rate of change of volume of a small moving 'blob' of fluid of fixed mass δm, by showing that the *relative* rate of expansion of the blob's volume equals the divergence of the velocity field.

3. For a 2D flow $\mathbf{u}(\mathbf{r}, t) = (u, v, 0)$, a *streamline* is defined as a curve that is in the direction of $\mathbf{u}$ at each point. (This concept is most useful when the flow – and the streamlines – are steady.)

Sketch the streamlines, and the flow directions for the following flows:

(a) $\mathbf{u} = (-\Omega y, \Omega x, 0)$ as in Q1, with $\Omega > 0$,
(b) $\mathbf{u} = (A, 0, 0)$, where $A > 0$,
(c) $\mathbf{u} = (A, 2A, 0)$, where $A > 0$,
(d) $\mathbf{u} = (\alpha x, -\alpha y, 0)$, where $\alpha > 0$.

Verify that each flow is incompressible, and evaluate the vorticity in each case.

4. The Lagrangian view of fluid flow is to track the trajectories of individual fluid 'particles'. Consider a particle whose coordinates at time t are given by

$$x = X(t) = X_0\, e^{\alpha t}\ , \qquad y = Y(t) = Y_0\, e^{-\alpha t}\ .$$

Calculate the velocity of this particle as a function of its position and show that, for varying values of X_0 and Y_0, the resulting flow is given in Eulerian form by the velocity field in Q3(d).

Find y as a function of x for the trajectory of a single particle; hence sketch this trajectory and verify that it coincides with a streamline as sketched qualitatively in Q3(d).

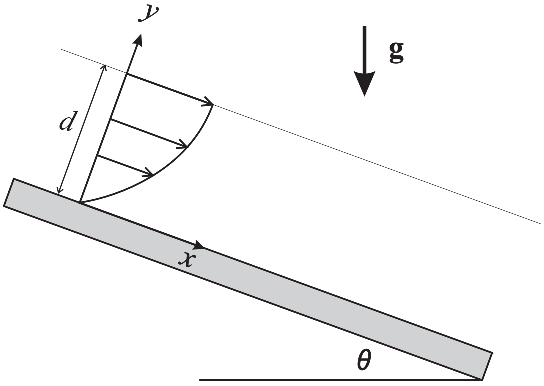
Give a qualitative description of the time evolution of the body of fluid that at $t = 0$ occupies the rectangle $|x| \le a$, $|y| \le b$.

5. Give a rough order-of-magnitude estimate of the Reynolds number for

   (a) flow past the wing of a jumbo jet flying at 150 m s^{-1},
   (b) a cup of tea being stirred,
   (c) a thick layer of treacle draining off a spoon,
   (d) a spermatozoan with tail of length 10 μm swimming at 0.1 mm s^{-1} in water.

Take the kinematic viscosity ν to be $10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ for water, $1.5 \times 10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ for air and $0.12\,\mathrm{m}^2\,\mathrm{s}^{-1}$ for treacle. Are you surprised that $\nu_{\text{water}} < \nu_{\text{air}}$?

6. Consider steady, 2D incompressible viscous flow down an inclined plane, under gravity. Choose axes as shown in the diagram, and assume that the velocity $\mathbf{u}$ depends only on y.



What are the boundary conditions for u and v at $y = 0$? Using incompressibility, show that $v = 0$ throughout the flow. Write down the x- and y-components of the Navier-Stokes equation, and from the y-component show that the pressure is of the form $p(x, y) = -\rho g y \cos\theta + F(x)$, where F is an arbitrary function. Assuming that the pressure at the free surface $y = d$ equals the constant atmospheric pressure p_0, show further that

$$p = p_0 + \rho g(d - y)\cos\theta\ .$$

From the x-component of the N-S equation, together with an appropriate boundary condition at $y = 0$ and the zero tangential stress condition $\mu du/dy = 0$ at the free surface $y = d$, show that

$$u = \frac{g}{2\nu}\, y(2d - y)\sin\theta\ .$$

Show that the volume flux per unit distance in the z-direction is $gd^3 \sin\theta/(3\nu)$.

7. The *circulation* Γ_{C} around a closed curve C lying within a fluid is defined as the line integral

$$\Gamma_{\text{C}} = \oint_{\text{C}} \mathbf{u} \cdot d\mathbf{l}\ .$$

Using Stokes' theorem from vector calculus, show that Γ_{C} may also be written as a surface integral

$$\Gamma_{\text{C}} = \int_{\text{S}} \boldsymbol{\omega} \cdot d\mathbf{S}\ ,$$

where $\boldsymbol{\omega}$ is the vorticity and S is a surface spanning C and lying wholly within the fluid region. What is the circulation if the flow is irrotational?

8. Consider inviscid swirling vortex flow around a solid cylinder of radius a, given by

$$\mathbf{u} = \frac{B}{r}\mathbf{i}_\theta \ , \quad r \geq a \ ,$$

where B is constant and non-zero. Show that the vorticity is zero everywhere within the fluid. Consider a circle C, concentric with the cylinder and lying outside it: show that the circulation Γ_C around this circle is $2\pi B$, and therefore non-zero. How do you reconcile this result with the last part of Q7?

*9. (Optional) Let $a \to 0$ in the previous question (i.e. the inner cylinder is absent). Show that in this case $\boldsymbol{\omega} = 2\pi B\delta(x)\delta(y)\,\mathbf{i}_z$, where δ is the Dirac delta-function and $\mathbf{i}_z$ is a unit vector in the z-direction. What is the magnetostatic analogue of this flow?

10. Consider a 2D inviscid, incompressible, irrotational flow with clockwise swirl, around a cylinder, as in Section 4.5 of the lecture notes. Starting with the expression for the pressure given there, show that the lift force L is given by

$$L = -\rho U \Gamma_\mathrm{C} \ ,$$

where Γ_C is the circulation around a circle concentric with the cylinder and lying outside it. Why is $\Gamma_\mathrm{C} < 0$ in this case?

Give a brief explanation of the relevance of this result to the physics of aircraft flight.

11. Write down the dispersion relation for linear surface waves on deep water. Hence show that the group velocity for such waves equals $gT/4\pi$, where T is the wave period.

Surface waves generated by a mid-Atlantic storm arrive at the British coast with period 15 seconds. A day later the period of waves arriving has dropped to 12.5 seconds. Roughly how far away did the storm occur?

12. To investigate how nonlinearity tends to produce wave-steepening, consider the model nonlinear equation

$$\frac{\partial \eta}{\partial t} + \alpha\eta\frac{\partial \eta}{\partial x} = 0 \ ,$$

where α is a positive constant. Verify that a solution of this equation is given by

$$\eta = f(x - \alpha\eta t) \ .$$

Note that this is an *implicit* relation for η, since η appears on both the left and right sides. [Hint: show that $\partial\eta/\partial x = f'/(1 + \alpha t f')$ and derive a similar equation for $\partial\eta/\partial t$.]

Consider a waveform f consisting of a single smooth hump. Show that parts of the hump with larger η travel faster than those with smaller η, and that one side of the hump therefore steepens with time.

13. (a) For steady, incompressible, very viscous flow (Stokes flow) the Navier-Stokes equation leads to the equation

$$\nabla^2(\nabla \times \mathbf{u}) = \mathbf{0} \, .$$

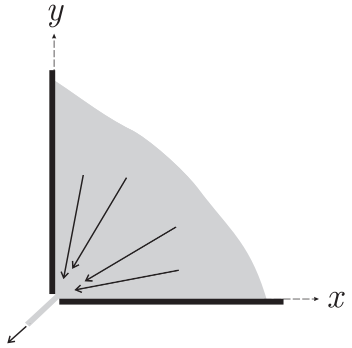
Give a clear physical statement of what is meant by 'very viscous' in this context, and state the corresponding condition on the Reynolds number.

(b) A two-dimensional Stokes flow in the xy-plane, independent of z, is described by a streamfunction ψ such that

$$\mathbf{u} = \nabla\psi \times \mathbf{k} \, ,$$

where $\mathbf{k}$ is a unit vector in the z-direction. Find an expression for the vorticity in Cartesian coordinates and hence (or otherwise) show that ψ satisfies the equation

$$\nabla^2 \left(\nabla^2 \psi\right) = 0 \, .$$



(c) The diagram shows a two-dimensional Stokes flow in which a very viscous fluid (indicated by shading) is being forced through a small gap at one edge of a rectangular box. Introduce plane polar coordinates r, θ, where $\theta = 0$ corresponds to the x-axis, and verify that the streamfunction

$$\psi = A\cos(2\theta) \, ,$$

where A is a constant, satisfies the governing equation. Derive expressions for the radial and azimuthal components of the corresponding velocity field. [You will need to look up the expressions for ∇^2 and grad in cylindrical polar coordinates.]

Give a brief description of this flow field and show that it satisfies the appropriate boundary conditions on the interior walls of the box.

(d) Derive a relation between the constant A and the rate of extraction Q of fluid volume from the box, per unit distance in the z-direction. Using a dimensional analysis involving relevant properties of the flow, including Q, find a dimensionless parameter that characterises the flow.

14. Show that the acceleration $D\mathbf{u}_R/Dt$ of a fluid element relative to a reference frame rotating at angular velocity Ω can be expressed in terms of the acceleration in an inertial frame $D\mathbf{u}_I/Dt$ as

$$\frac{D\mathbf{u}_I}{Dt} = \frac{D\mathbf{u}_R}{Dt} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{u}_R.$$

Briefly discuss the physical interpretation of the 2nd and 3rd terms on the RHS of this equation and the effect they have on the flow.

15. (a) Show that the flow relative to the rotating frame in a rapidly rotating fluid system approximately satisfies the geostrophic relation

$$2\boldsymbol{\Omega} \times \mathbf{u}_R = -\frac{1}{\rho}\nabla p$$

and outline the conditions for this balance to dominate.

(b) Define the Rossby number, and estimate its value for

(i) the Earth's liquid core, where flow speeds $\sim 1 - 10$ mm s^{-1} occur,
(ii) a gas turbine jet engine, rotating at 10^4 rpm with throughflow velocities of 300 m s^{-1},
(iii) a tornado, with tangential speeds of 150 m s^{-1} and a diameter of 200 m.

(c) Estimate how much higher the water level is on the French coast than on the English coast when there is an eastward flow through the English Channel at a typical speed of 1 m s^{-1}. You may assume that the English Channel is approximately 30 km wide.

(d) Show that

$$(2\boldsymbol{\Omega}.\nabla)\mathbf{u}_R \simeq 0$$

in a homogeneous rotating fluid, and discuss the implications of this result.

16. The motion in a viscous boundary layer at the top free surface of a uniformly rotating fluid satisfies the linearized equations

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2},$$

$$\frac{\partial v}{\partial t} + 2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2},$$

where $\Omega(> 0)$ and ν are constants.

(a) Give a brief account of the assumptions underlying these equations, indicating how they are obtained from the full Navier–Stokes equations, and explain why only vertical derivatives are used in the last terms on the right-hand side.

(b) Consider a system where the deep geostrophic flow is negligible. A steady wind blows across the surface of the water along the $+x$ direction, leading to a constant surface stress $\tau = (\tau_x, 0)$ where τ_x is the x component of the surface stress. Given that the water depth H is much greater than $h = (\nu/\Omega)^{1/2}$, and by considering the boundary conditions at

$z = 0$ (surface) and $z \to -\infty$ on u and v, show that the horizontal velocities are of the form

$$u = A\exp(Kz)(\sin Kz + \cos Kz),$$

$$v = A\exp(Kz)(\sin Kz - \cos Kz),$$

and obtain expressions for the constants A and K in terms of τ, Ω and ν. [Hint: use a change of variable $Z = u + iv$]

(c) Sketch the form of (u, v) as a function of depth z, clearly labelling the direction of the surface wind stress and other significant features.

(d) By integrating these equations for (u, v) with respect to z from the surface into the deep interior of the flow, obtain expressions for the horizontal Ekman transport (U_E, V_E) and comment on its direction relative to τ.

(e) The boundary layer in the Earth's oceans is observed to have a depth of approximately 100 m at latitude 65° N (where Ω can be assumed to take the value 7×10^{-5} s^{-1}). Hence estimate the order of magnitude of ν (representing a *turbulent eddy viscosity*). A steady wind blows from the west, leading to a surface stress of 0.1 N m^{-2}. A large iceberg extends to a depth of 200 m beneath the ocean surface. By assuming that the motion of the iceberg follows the Ekman transport in the ocean boundary layer, estimate the speed and direction at which the iceberg drifts.